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ON THE DYNAMIC CONTACT PROBLEM FOR A HALF-PLANE REINFORCED BY A FINITE ELASTIC STRIP

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The dynamic contact problem for a half-plane reinforced on its boundary by a finite elastic strip is considered. The solution of the problem reduces to solving an integral equation of the first kind, and then an infinite system of linear equations by using Chebyshev polynomials. It is proved that this infinite system of equations is quasi-completely regular. Moreover, a simple analytical expression, completely admissible for practical applications and differing by an arbitrarily small amount from the exact expression, is obtained for the kernel of the integral equation. In this case, for definite values of some physical parameter, the complete regularity of the appropriate infinite system of equations is proved in addition to the quasi-complete regularity and numerical results are obtained for the law of variation of the amplitude of the tangential contact stresses under the strip.

The problem under consideration is related to problems of load transfer from stringers to elastic solids which are important for engineering practice. The case of an infinite or semi-infinite strip has been examined earlier [1].

1. Formulation of the problem. Derivation of the governing equation. Let a semi-infinite plane be reinforced by an elastic strip of constant sufficiently small thickness h welded to a finite segment of its boundary $[-a, a]$. The purpose of this paper is to determine the contact stress distribution law along the segment connecting the elastic strip to the half-plane when a concentrated horizontal harmonic force $P \sin \omega t$ (Fig. 1) is applied to one of the strip ends. For simplicity in the computations, we shall henceforth take this force as $Pe^{-i\omega t}$ (it is hence evidently necessary to take the imaginary part of the solution with the reverse sign). As in

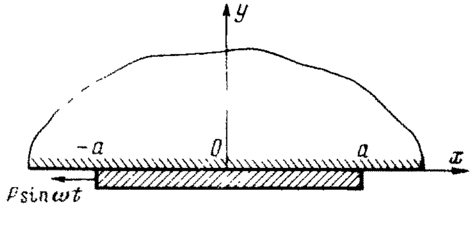

Fig. 1

[2, 3], let us assume that the strip bending stiffness is negligibly small because of the smallness of the thickness h, and hence, the normal strip pressure on the half-plane can be neglected (model of a one-dimensional elastic strip continuum). This permits considering only tangential contact stresses to act under the strip, i. e. the strip is in a uniaxial state of stress.

The strip vibrations equation under the mentioned assumptions is [1]

$$\frac{\partial^2 u^{(1)}(x,t)}{\partial x^2} - \frac{\rho_1}{E_1}\frac{\partial^2 u^{(1)}(x,t)}{\partial t^2} = -\frac{1}{E_1 h}\tau(x,t), \quad -a < x < a$$

Here $u^{(1)}(x, t)$ is the horizontal displacement of points of the strip, ρ_1 is the density of the strip material, $\tau(x, t)$ is the unknown contact stress at the point x at the instant t acting on the strip along the line connecting it to the half-plane, and E_1 is the strip elastic modulus.

Considering steady-state strip vibrations, let us assume

$$u^{(1)}(x, t) = u_*(x) e^{-i\omega t}, \qquad \tau(x, t) = \tau_*(x) e^{-i\omega t}$$

Furthermore, taking into account the value of the axial stress at the ends of the strip, we obtain a boundary value problem to determine the amplitude of the displacements, whose solution is given by the formula

$$u_*^{(1)}(ax) = -\frac{a^2}{E_1 h}\int_{-1}^{1} G(kax, kas)\,\tau_*(as)\,ds + \frac{P}{E_1 h}\frac{\cos ka(x-1)}{k\sin 2ka} \quad (1.1)$$

$$-1 \leqslant x \leqslant 1, \quad k = \omega\sqrt{\rho_1/E_1}$$

$$G(kax, kas) = \begin{cases} -\dfrac{1}{ak\sin 2ka}\cos[ka(x-1)]\cos[ka(s+1)], & x \leqslant s \\ & -1 \leqslant x, s \leqslant 1 \\ -\dfrac{1}{ak\sin 2ka}\cos[ka(s-1)]\cos[ka(x+1)], & x \geqslant s \end{cases}$$

after the transition has been made from the segment $[-a, a]$ to the segment $[-1, 1]$ with which we shall henceforth deal. Here $aG(kax, kas)$ is the Green's function for the same problem but with zero boundary conditions [4]. On the other hand, the amplitude of the horizontal displacements of the boundary points of the elastic half-plane due to the horizontal harmonic force of amplitude $\tau_*(x)$ is given by the formula [1]

$$u_*^{(2)}(ax) = \frac{a}{\mu_2}\int_{-1}^{1} K(k_2 a|x-s|)\,\tau_*(as)\,ds, \quad k_2 = \omega\sqrt{\rho_2/\mu_2} \quad (1.2)$$

$$K(z) = \frac{1}{2\pi}\int_{-1}^{\infty} k(s)\,e^{-izs}ds, \qquad k(s) = \frac{\sqrt{s^2-1}}{(2s^2-1)^2 - 4s^2\sqrt{(s^2-1)(s^2-\varepsilon^2)}}$$

$$\varepsilon = \sqrt{\frac{\mu_2}{\lambda_2 + 2\mu_2}} = \sqrt{\frac{1-2\nu}{2(1-\nu)}}$$

Here λ_2, μ_2 are Lamé parameters, ρ_2 is the density, and ν is the Poisson's ratio of the half-plane material.

Let us note that the last integral should be understood in the Cauchy principal value sense since the integrand has a first order pole on the real axis [1, 5]. Furthermore, the condition $u_*^{(1)}(ax) = u_*^{(2)}(ax)$, $-1 \leqslant x \leqslant 1$ should be satisfied on the segment connecting the strip to the half-plane, and which in combination with (1.1) and (1.2) reduces the problem of determining the amplitude of the tangential contact stresses to solving an integral equation of the first kind

$$\int_{-1}^{1} [K(k_2^*|x-s|) + \lambda^* G(k^*x, k^*s)]\,\varphi(s)\,ds = \frac{\cos[k^*(x-1)]}{k^*\sin 2k^*} \qquad (1.3)$$

$$\varphi(s) = \frac{\tau^*(as)\,a}{\lambda^* P}, \quad \lambda^* = \frac{\mu_2 a}{E_1 h}, \quad k_2^* = ak_2, \quad k^* = ak$$

To investigate the structure of the function $K(z)$, let us note that its Fourier transform $k(s)$ has the expansion

$$k(s) = -\frac{1}{2(1-\varepsilon^2)}\frac{1}{|s|} - \frac{1}{8}\frac{1+\varepsilon^4}{(1-\varepsilon^2)^2}\frac{1}{s^2|s|} + \frac{\varepsilon^8 - 2(\varepsilon^6+\varepsilon^4+\varepsilon^2) - 3}{32(1-\varepsilon^2)}\frac{1}{s^4|s|} + k_1(s)$$

$$k_1(s) = k(s) + \frac{1}{2(1-\varepsilon^2)}\frac{1}{|s|} + \frac{1}{8}\frac{1+\varepsilon^4}{(1-\varepsilon^2)^2}\frac{1}{s^2|s|} - \frac{\varepsilon^8 - 2(\varepsilon^6+\varepsilon^4-\varepsilon^2)^{-3}}{32(1-\varepsilon^2)}\frac{1}{s^4|s|}$$

in the neighborhood of the infinitely distant point. The function $k_1(s)$ at the zero point evidently has a nonintegrable singularity. Because of known formulas [6], the function

$$k_*(s) = \iiiint\!\!\int_{-\infty}^{s} k_1(\tau)\,d\tau$$

is however integrable at this same point. Because $k_*(s) \sim \text{const}/s\,|s|$ as $|s| \to \infty$, then $k_*(s)$ will be integrable on the whole real axis. Moreover, the formula

$$F(k_1(s)) = iz^5 F^{-1}(k_*(s))$$

holds, where F is the Fourier transform in the sense of the theory of generalized functions, and F^{-1} is the inverse transform [6, 7]. Taking this last formula into account, we obtain the following representation (C is still an unknown constant):

$$K(k_2^*x) = -\frac{1}{2(1-\varepsilon^2)}\frac{1}{\pi}\ln\frac{1}{|k_2^*x|} - \frac{C}{2(1-\varepsilon^2)} - \frac{1}{16}\frac{1+\varepsilon^4}{(1-\varepsilon^2)^2}(k_2^*x)^2 \times \left(\frac{1}{\pi}\ln|k_2^*x| + C\right) + \frac{3}{32\pi}\frac{1+\varepsilon^4}{(1-\varepsilon^2)^2}(k_2^*x)^2 + \frac{\varepsilon^8 - 2(\varepsilon^6+\varepsilon^4-\varepsilon^2) - 3}{768(1-\varepsilon^2)} \times (k_2^*x)^4\left(\frac{1}{\pi}\ln\frac{1}{|k_2^*x|} + C + \frac{25}{12\pi}\right) + \frac{i(k_2^*x)^5}{2\pi}\int_{-\infty}^{\infty} k_*(s)\,e^{-isk_2^*x}\,ds \qquad (1.4)$$

It should be noted that another representation of the function $K(k_2^*x)$ of the same structure but containing any finite number of terms in even powers of the argument k_2^*x can be obtained completely analogously in place of the representation (1.4). Proceeding to determine the value of the constant C, let us note that the function $K(k_2^*, x)$ can also be represented as ($\mathrm{ci}\,(k_2^*x)$ is the cosine integral)

$$K(k_2^*x) = \frac{1}{\pi}\int_0^1 k(s)\cos(k_2^*xs)\,ds + \int_1^{\infty}\left[k(s) + \frac{1}{2(1-\varepsilon^2)}\frac{1}{s}\right] \times \cos(k_2^*xs)\,ds + \frac{\mathrm{ci}\,(k_2^*x)}{2(1-\varepsilon^2)}$$

There follows from these two representations of the function $K\ (k_2{}^*x)$ that the constant C not yet known is expressed by the formula

$$C = -2(1-\varepsilon^2)\lim_{x\to 0}\left[K\ (k_2{}^*x) - \frac{1}{2(1-\varepsilon^2)\pi}\ln|k_2{}^*x|\right] = \tag{1.5}$$

$$-\frac{1}{\pi}\ln\gamma - \frac{2(1-\varepsilon^2)}{\pi}\int_0^1 k(s)\,ds - \frac{2(1-\varepsilon^2)}{\pi}\int_1^\infty\left[k(s)+\frac{1}{2(1-\varepsilon^2)}\frac{1}{s}\right]ds$$

Therefore, on the basis of (1.4) we have the following representation:

$$K(k_2{}^*x) = -\frac{1}{2(1-\varepsilon^2)}\frac{1}{\pi}\ln\frac{1}{|k_2{}^*x|} + R(k_2{}^*x)$$

where the function $R\ (k_2{}^*x)$ possesses the property that its second derivative is square summable on the segment $[-1, 1]$. The function $K\ (k_2{}^*x)$ in this formula is represented as the sum of its principal and regular parts. Furthermore we have ($\theta(x)$ is the Heaviside function)

$$\frac{\partial G(k^*x, k^*s)}{\partial x} = G^*(k^*x, k^*s) + \theta(x-s) \tag{1.6}$$

$$G^*(k^*x, k^*s) = \begin{cases} \dfrac{1}{\sin 2k^*}\sin[k^*(x-1)]\cos[k^*(s+1)], & x \leqslant s \\ \dfrac{1}{\sin 2k^*}\cos[k^*(s-1)]\sin[k^*(x+1)] - 1, & x \geqslant s \end{cases}$$

This last function is continuous in the square $-1 \leqslant x,\ s \leqslant 1$, and has continuous partial derivatives with respect to the variables x and s in this same square. Taking account of (1.4) and (1.6) and differentiating both sides of (1.3), we obtain the following singular integral equation of the first kind in the unknown function $\varphi(s)$:

$$\int_{-1}^1\left\{\frac{1}{\pi}\frac{1}{s-x} - 2(1-\varepsilon^2)\frac{\partial R(k_2{}^*|x-s|)}{\partial x} - 2(1-\varepsilon^2)\lambda^* G^*(k^*x, k^*s) - \right. \tag{1.7}$$

$$\left. 2(1-\varepsilon^2)\lambda^*\theta(x-s)\right\}\varphi(s)\,ds = 2(1-\varepsilon^2)\,\frac{\sin[k^*(x-1)]}{\sin 2k^*}$$

The solution of the dynamic contact problem for a half-plane reinforced on a finite part of its boundary by a finite strip of small thickness therefore reduces to solving a singular integral equation. The kernel og this equation consists of two members, the first cf which is a Cauchy kernel, and the other is a square summable function in the square $-1 \leqslant x,\ s \leqslant 1$.

2. Reduction of (1.7) to an infinite system of linear equations.

Following [8], let us seek the solution of (1.7) in the form

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\left(a_0 + \sum_{n=1}^\infty a_n T_n(x)\right), \quad T_n(x) = \cos(n\arccos x),\ -1 < x < 1 \tag{2.1}$$

where $T_n\ (x)$ are Chebyshev polynomials of the first kind. Substituting $\varphi\ (x)$ from (2.1) into (1.7) and using the relationship [9]

$$\frac{1}{\pi}\int_{-1}^1\frac{T_n(s)\,ds}{(s-x)\sqrt{1-s^2}} = U_{n-1}(x), \qquad U_{n-1}(x) = \frac{\sin(n\arccos x)}{\sin(\arccos x)}$$

$$n = 1, 2, 3, \ldots; \qquad -1 \leqslant x \leqslant 1$$

where $U_{n-1}(x)$ are Chebyshev polynomials of the second kind, we obtain the following infinite system of linear equations in the coefficients $\{a_n\}_{n=1}^{\infty}$: by a known method:

$$a_m + \frac{4(1-\varepsilon^2)}{\pi}\lambda^* \sum_{n=1}^{\infty} K_{m,n} a_n + \frac{2}{\pi}\sum_{n=1}^{\infty} R_{m,n} a_n = a_0 \varphi_m + f_m \qquad (2.2)$$

Here

$$K_{m,n} = \begin{cases} \dfrac{2m[1+(-1)^{n-m}]}{[n^2-(m-1)^2][n^2-(m+1)^2]}, & n \neq m-1, \quad n \neq m+1 \\ 0, & n = m-1, \quad n = m+1 \end{cases}$$

$$R_{m,n} = \int_{-1}^{1}\int_{-1}^{1} \frac{R_1(x,s)T_n(s)}{\sqrt{1-s^2}}\, ds \sqrt{1-x^2}U_{m-1}(x)\,dx, \quad m, n = 1, 2, \dots$$

$$R_1(x,s) = -2(1-\varepsilon^2)\frac{\partial R(k_2^*|x-s|)}{\partial x} - 2(1-\varepsilon^2)\lambda^* G^*(k^*x, k^*s)$$

$$\varphi_m = \frac{4}{\pi}(1-\varepsilon^2)\lambda^* b_m - \frac{2}{\pi}R_{m,0}$$

$$b_m = \frac{\pi^2}{4}, \quad m = 1, \qquad b_m = \frac{2m[1-(-1)^m]}{(m^2-1)^2}, \quad m \neq 1$$

$$f_m = \frac{4(1-\varepsilon^2)}{\pi \sin 2k^*}\int_{-1}^{1} \sin[k^*(x-1)]\sqrt{1-x^2}U_{m-1}(x)\,dx, \qquad m = 1, 2, \dots$$

3. Investigation of the infinite system (2.2). An investigation of the infinite system (2.2) in the case of just one kernel $K_{m,n}$ is contained in [8]. Addition of the kernel $R_{m,n}$ to this kernel does not destroy the regularity of the original infinite system in the sense of its quasi-complete regularity. Indeed, on the basis of the above mentioned properties of the function $R_1(x, s)$ we can write

$$R_{m,n} = \frac{1}{h} h_{m,n}, \qquad m, n = 1, 2, \dots$$

$$h_{m,n} = \int_{-1}^{1}\int_{-1}^{1} R_0(x,s)\sqrt{1-s^2}\sqrt{1-x^2}\,U_{n-1}(s)\,u_{m-1}(x)\,dx\,ds, \quad R_0(x,s) = \frac{\partial R_1(x,s)}{\partial s}$$

Then

$$S_m = \sum_{n=1}^{\infty} |R_{m,n}| = \sum_{n=1}^{\infty} \frac{1}{n}|h_{m,n}|, \qquad m = 1, 2, \dots$$

Now, noting that the coefficients $\{h_{m,n}\}_{m,n=1}^{\infty}$ are square summable Fourier coefficients in the square $-1 \leqslant x, s \leqslant 1$ of the function $R_0(x, s)$ in the complete orthogonal system of functions $\{U_{n-1}(s)\,U_{m-1}(x)\}_{m,n=1}^{\infty}$, we can assert on the basis of the Bessel inequality that

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |h_{m,n}|^2 < \infty$$

then the series

$$\sum_{m=1}^{\infty} C_m, \qquad C_m = \sum_{n=1}^{\infty} |h_{m,n}|^2$$

also converges by the theorem known from analysis [10]. Therefore, at least

$$C_m = O\left(\frac{1}{m^{1+\delta}}\right) \quad \text{for} \quad m \to \infty \tag{3.1}$$

where δ is a small positive number. Furthermore

$$S_m \leqslant \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^2}} \sqrt{\sum_{n=1}^{\infty} |h_{m,n}|^2} = \frac{\pi}{\sqrt{6}} C_m$$

Taking (3.1) into account we have

$$S_m = O\left(\frac{1}{m^{\frac{1+\delta}{2}}}\right), \quad m \to \infty$$

which indeed proves the assertion expressed above concerning the quasi-complete regularity of the infinite system.

Now, let us note that to determine a_0, we should substitute $x = -1$ in the expression obtained after substituting (2.1) into (1.7). We obtain

$$\sum_{n=1}^{\infty} d_n a_n = a_0 D - 2(1-\varepsilon^2) \tag{3.2}$$

Here

$$d_n = n(-1)^n + \int_{-1}^{1} \frac{R_1(-1,s)\, T_n(s)}{\sqrt{1-s^2}} ds, \qquad D = -\int_{-1}^{1} \frac{R_1(-1,s)}{\sqrt{1-s^2}} ds$$

After the coefficients $\{a_n\}_{n=1}^{\infty}$ have been determined, the unknown coefficient a_0 will be determined from (3.2).

4. Case of the small parameter k_2^*. In this case, neglecting terms on the order of $(k_2^*)^4 \ln k_2^*$, we have in the representation (1.4)

$$K(k_2^* x) = -\frac{1}{2(1-\varepsilon^2)} \frac{1}{\pi} \ln \frac{1}{|k_2^* x|} - \frac{C}{2(1-\varepsilon^2)} - \alpha_0 \left(\frac{k_2^* x}{4}\right)^2 \left(\frac{1}{\pi} \ln |k_2^* x| + C\right) + \frac{3\alpha_0}{32\pi}(k_2^* x)^2, \quad \alpha_0 = \frac{1+\varepsilon^4}{(1-\varepsilon^2)^2} \tag{4.1}$$

Then (1.7) goes over into the following:

$$\int_{-1}^{1} \left[\frac{1}{\pi} \frac{1}{s-x} + \alpha_1 (x-s) \ln |x-s| + \alpha_2 (x-s) - 2(1-\varepsilon^2)\lambda^* G(x,s) - 2(1-\varepsilon^2)\lambda^* \theta(x-s)\right] \varphi(s)\, ds = \frac{2(1-\varepsilon^2)}{\sin 2k^*} \sin [k^*(x-1)]$$

where

$$\alpha_1 = \frac{1}{4\pi} \frac{1+\varepsilon^4}{1-\varepsilon^4} (k_2^*)^2, \qquad \alpha_2 = \alpha_1 (\pi C + \ln k_2^* - 1)$$

Again representing the solution of this equation in the form (2.1), we analogously obtain the following infinite system of linear algebraic equations relative to the coefficients $\{a_n\}_{n=1}^{\infty}$:

$$a_m + \frac{4(1-\varepsilon^2)}{\pi}\lambda^* \sum_{n=1}^{\infty} K_{m,n}a_n + \frac{2}{\pi}\sum_{n=1}^{\infty} K^*_{m,n}a_n + \tag{4.2}$$
$$\frac{4(1-\varepsilon^2)\lambda^* k^*}{\pi \sin 2k^*}\sum_{n=1}^{\infty} R^*_{m,n}a_n = a_0\varphi_m{}^* + f_m$$

$$R^*_{m,n} = -\frac{1}{n}\frac{\sin 2k^*}{k^*}\int_{-1}^{1}\int_{-1}^{1}\frac{\partial G^*(x,s)}{\partial s}\sqrt{1-s^2}U_{n-1}(s)\,ds\sqrt{1-x^2}U_{m-1}(x)\,dx$$

$$K^*_{m,n} = \begin{cases} \left.\begin{array}{l} 0, \quad n \neq m, \quad n \neq m+2, \quad n \neq m-2 \\ \dfrac{\alpha_1\pi^2}{4}\dfrac{1}{n^2-1}, \quad n = m \\ -\dfrac{\alpha_1\pi^2}{8}\dfrac{1}{n(n-1)}, \quad n = m+2 \\ -\dfrac{\alpha_1\pi^2}{8}\dfrac{1}{n(n+1)}, \quad n = m-2 \end{array}\right\} n \neq 1 \\ \left.\begin{array}{l} 0, \quad m \neq 1, \quad m \neq 3 \\ \dfrac{\alpha_1\pi^2}{4}\ln 2 - (\alpha_1+\alpha_2)\dfrac{\pi^2}{4} + \alpha_1\dfrac{\pi^2}{16}, m = 1 \\ -\dfrac{\alpha_1\pi^2}{16}, \quad m = 3 \end{array}\right\} n = 1 \end{cases}$$

$$\varphi_m{}^* = \frac{4}{\pi}(1-\varepsilon^2)\lambda^*(b_m + q_m) - e_m$$

Here

$$e_m = \begin{cases} \dfrac{\pi}{2}[\alpha_1(\ln 2 - 1) + \alpha_2], & m = 2 \\ 0, & m \neq 2 \end{cases}$$

$$q_m = \frac{(k^*)^3}{\sin 2k^*}\begin{cases} \gamma_m - \dfrac{96m[(-1)^m+1]}{(m^2-9)^2(m^2-1)^2}, & m \neq 1, \quad m \neq 2, \quad m \neq 3 \\ \gamma_1 - \dfrac{3\pi^2}{16}, & m = 1 \\ \gamma_2 + \dfrac{13\pi^2}{16} - \dfrac{128}{175}, & m = 2 \\ \gamma_3 + \dfrac{\pi^2}{16}, & m = 3 \end{cases}$$

$$\gamma_m = \frac{m\pi^2\cos 2k^*}{(k^*)^4}J_m(k^*)\cos\left(k^* + \frac{m\pi}{2}\right)$$
$$f_m = \frac{4(1-\varepsilon^2)}{k^*\sin 2k^*}J_m(k^*)\cos\left(k^* + \frac{m\pi}{2}\right) \qquad m = 1, 2, \ldots$$

($J_m(x)$ is the Bessel function of the first kind).

Proceeding completely analogously to the previous Sect., we find that the system of infinite equations under consideration is completely regular under the condition

$$\frac{12\lambda^*}{\pi} + \frac{4\lambda^* k^*}{\sqrt{6}\sin 2k^*} + r < 1$$

where

$$r = \frac{\alpha_1 \pi}{2} \max\left\{\left|\ln 2 + \frac{1}{4} - \ln k_2{}^* - \pi C\right| + \frac{1}{12}; \frac{2}{5}\right\}$$

When the strip becomes a stamp, i.e. for $E_1 = \infty$, the corresponding infinite system of linear equations becomes

$$b_{2j} + \frac{2}{\pi}\sum_{i=1}^{\infty} K_{2j,\,2i} b_{2i} = b_0 e_{2j}, \quad j = 1, 2, \ldots \tag{4.3}$$

and the solution of the truncated system with an arbitrary number of unknowns N is determined by the formula

$$b_{2j} = B_{2j} e_2 b_0, \qquad j = 1, 2, \ldots, N$$

Here

$$B_{2j} = (-1)^{j+1} \frac{\prod\limits_{\rho=0}^{j-1} K^*_{2(j-\rho),\,2(j-\rho)}}{\prod\limits_{n=0}^{j-1} [d_0^{(j-1-n)}; d_1^{(n)}, d_2^{(n)}, \ldots, d_{N+n-j}^{(n)}]}, \qquad j = 1, 2, \ldots, N-1$$

$$B_{2N} = -\frac{K^*_{2N,\,2N-2}}{K^*_{2N,\,2N}} B_{2N-2}, \qquad K^*_{2,\,0} = 1$$

$[d_0^{(j-1-n)}; d_1^{(n)}, \ldots, d_{N+n-j}^{(n)}]$ — $N + n - j$-term continued fraction)

$$d_{1+q} = -\frac{K_{2(j+q-n+1),\,2(j+q-n+1)}}{\prod\limits_{l=j-n+1}^{j+q-n+1} (K^*_{2l,\,2l-2}) K^*_{2l-2,\,2l}}, \qquad d_0^{(j-1-n)} = -K^*_{2(j-n),\,2(j-n)}$$

$$q = 0, 1, \ldots, N-1+n-j; \quad j = 1, 2, \ldots, N-1; \quad n = 0, 1, \ldots, j-1$$

$$b_0 = \frac{\pi P}{M\omega^2 L} \qquad (M \text{ — mass of the stamp})$$

$$L = \int_{-1}^{1}\left[\int_{-1}^{1} \frac{K(k_2{}^*|x-s|)}{\sqrt{1-s^2}} ds + e_2 \sum_{j=1}^{N} B_{2j} \int_{-1}^{1} K(k_2{}^*|x-s|) \frac{T_{2j}(s)}{\sqrt{1-s^2}} ds\right] \times \frac{dx}{\sqrt{1-x^2}} + \frac{\pi^2}{M\omega^2}$$

Therefore, the solution of the problem posed in the case of a stamp will again reduce to the solution of the infinite system of equations (4.3) under the assumption made. However, in contrast to the case of a strip, the solution of the corresponding truncated system of equations with the arbitrary number of unknowns is successfully constructed in closed form. Moreover, this result holds even when any finite number of terms in even powers of $k_2{}^*x$ is retained in the representations (1.4).

Let us note that for simplicity of the calculations, only terms of the order of $(k_2{}^*)^2$ were retained in (1.4) although the proposed procedure for solving the integral equation (1.7) is applicable even when any finite number of terms in powers of the parameter $k_2{}^*$ is retained in this representation. In the majority of practical cases [5] $0 < k_2{}^* \leqslant 0.25$, and therefore, the approximation (4.1) made deviates slightly from the true value and corresponds sufficiently well to reality.

Let us turn to a description of the numerical results. Since the magnitude of the amplitude of the tangential contact stresses under the strip depends essentially on the value of the constant $C = C_1 + iC_2$, the value of this constant was first calculated on the

"Nairi-2" electronic computer by using (1.5). The values of the real and imaginary parts of this constant are presented below for different values of the physical parameter ε which depends, in turn, on the Poisson's ratio in the above-mentioned manner: $\varepsilon = 0.6416,\ 0.5774,\ 0$; $C_1 = 0.0094,\ 0.1262,\ 0.2261$; $C_2 = 0.6595,\ 0.7019,\ 0.7537$.

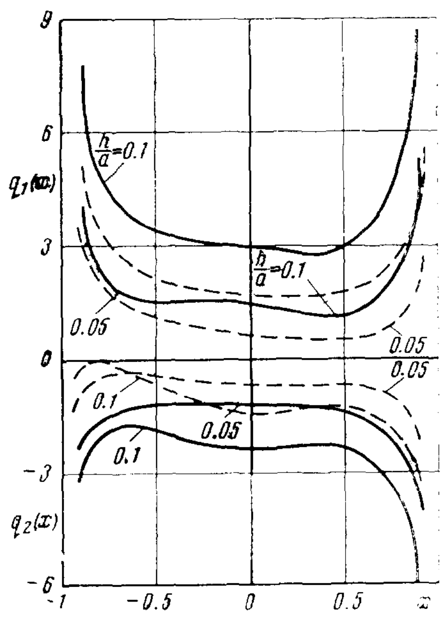

Fig. 2

For subsequent calculations it was assumed that $\rho_1 = \rho_2$ and $k_2^* = 0.25$. Then the complete regularity condition for the infinite system of equations (4.2) is satisfied for the mentioned values of the constant C as the parameter k^* varies within the interval (0, 0.022). The value $k^* = 0.02$ was taken for the calculations. Since

$$k^* = \sqrt{\frac{\rho_1}{E_1}}\,\omega a, \qquad k_2^* = \sqrt{\frac{\rho_2}{E_2}}\,\omega a$$

then the relationship

$$E_2 / E_1 = 0.0128\,(1 + \nu)$$

should hold between the materials of the strip and the base for values taken for the parameters k_2^* and k^*. Hence, it follows that the condition $E_1 > E_2$ must be conserved, which indeed agrees with the strip model assumed [11].

The truncated system of equations with ten unknowns, obtained from the infinite system (4.2), was then solved for given values of the mentioned parameters and for values of the ratio $h / a = 0.1$ and 0.05. The approximate expression

$$\frac{\partial G^*(x, s)}{\partial s} = -\frac{(k^*)^3}{\sin 2k^*} \times \begin{cases} xs + (x - s) - 1, & x \leqslant s \\ xs - (x - s) - 1, & x \geqslant s \end{cases}$$

was hence taken for the function $\partial G^*(x, s) / \partial s$.

The truncated system was preliminarily represented as

$$a_m + \frac{4(1-\varepsilon^2)}{\pi}\lambda^* \sum_{n=1}^{10} K_{m,n} a_n + \frac{2}{\pi}\sum_{n=1}^{10} K^*_{m,n} a_n + \tag{4.4}$$
$$\frac{4(1-\varepsilon^2)\lambda^* k^*}{\pi \sin 2k^*}\sum_{n=1}^{10} R^*_{m,n} a_n = a_0\varphi^*_m + f_m, \quad m = 1, 2, \ldots$$

If $\{a^{(1)}_m\}_{m=1}^{10}$ and $\{a^{(2)}_m\}_{m=1}^{10}$ are solutions of the system (4.4) for the right sides $\{f_m\}_{m=1}^{10}$ and $\{\varphi_m\}_{m=1}^{10}$, respectively, then the solution of the initial system (4.2) is expressed by the formula

$$a_m = a^{(1)}_m + a_0 a^{(2)}_m, \qquad m = 1, 2, \ldots \tag{4.5}$$

Substituting these expressions for the coefficients $\{a_m\}_{m=1}^{10}$ into (3.2) we find the coefficient a_0. When we know the coefficient a_0, the coefficients $\{a_m\}_{m=1}^{10}$ which are calculated from (4.5), will also be known.

We evidently have a system of algebraic equations with twenty unknowns in the real and imaginary parts of the coefficients $\{a_m\}_{m=1}^{10}$, $i = 1, 2$. These systems were solved on the "Nairi-2" electronic computer.

On the basis of the numerical values of the coefficients mentioned, a graph was constructed on the change in the real and imaginary parts of the function $\varphi'(x) = q_1(x) + iq_2(x)$ as a function of the values given above for the geometric and physical constants of the strip and the base.

Let us note that the true distribution law of the tangential contact stresses under a strip is given by the formula

$$\tau(x, t) = \frac{\lambda^* p}{a}[q_1(x)\sin\omega t - q_2(x)\cos\omega t]$$

Shown in Fig. 2 is the behavior of the change in the functions $q_1(x)$ and $q_2(x)$ as a function of the Poisson's ratio ν and the ratio h / a (the solid lines correspond to $\nu = 0.5$, the dashes to $\nu = 0.25$). Comparison of these graphs shows that for fixed values of ν a larger value of these functions corresponds to the small value of the ratio a / h. In this connection, let us note that $\lambda^* = 0.0064\, a / h$ for the values taken for the parameters k^* and k_2^*. When $\lambda^* = 0$, we have the case of a stamp, and when $\lambda^* \neq 0$, we have the case of a strip, and a small value of the elastic modulus of the strip material will correspond to a large value of this parameter for a fixed thickness h and length $2a$. A diminution in the strip elastic modulus can be interpreted as an increase in the parameter λ^*. The parameter λ^* grows with distance from the stamp, and absolute values of the tangential contact stresses have a tendency to decrease under the effect of the same force. As follows from the above, this same tendency is characteristic even for the behavior of the chage in the amplitude functions $q_1(x)$ and $q_2(x)$.

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METHOD OF ORTHOGONAL POLYNOMIALS IN PLANE ANTISYMMETRIC MIXED PROBLEMS OF ELASTICITY THEORY WITH TWO CONTACT SECTIONS

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The possibility is shown of applying the method of orthogonal polynomials to solve some integral equations of a special kind if the eigenfunctions of the integral operator corresponding to the principal (singular) part of the kernel are unknown. Use of the classical scheme [1 – 3] is impossible in this case. However, by using modified Chebyshev polynomials, an integral equation of the form

$$\int_k^1 \varphi(\xi) \ln \left| \frac{\xi + x}{\xi - x} \right| d\xi = \pi f(x) - \int_k^1 \varphi(\xi) G(\xi, x, \lambda)\, d\xi \tag{0.1}$$

$$G(\xi, x, \lambda) = \xi x G_*(\xi, x, \lambda), \quad k \leqslant x \leqslant 1, \quad \lambda \in (0, \infty), \quad k \in (0, 1)$$

is successfully reduced to an infinite algebraic system of the first kind convenient for approximate solution. Here λ, k are dimensionless parameters, G_* is a continuous, even, and symmetric function in ξ, x. Plane antisymmetric mixed problems of elasticity theory with two contact sections, odd in x, reduce to equations of the type (0.1). The odd function $f(x)$ describes the shape of the boundary layer on the contact section $k \leqslant |x| \leqslant 1$ altered under the effect of stamps.

Considered as an illustration is the problem of impressing two flat stamps into a strip.

1. Representing the function $f(x) = f_0(x) + \beta \operatorname{sgn} x$, we seek the solution $\varphi(\xi)$ of (0.1) as

$$\varphi(\xi) = \varphi_0(\xi) + \varphi_1(\xi) \tag{1.1}$$

$$\int_k^1 \varphi_0(\xi) \ln \left| \frac{\xi + x}{\xi - x} \right| d\xi = \pi f_0(x) \qquad (k \leqslant x \leqslant 1) \tag{1.2}$$

Here $\varphi_0(\xi)$, the solution of the integral equation (1.2), is given by formulas in [4] in which it is assumed that $x / a = x$, $\xi / a = \xi$, $b / a = k$, $a = 1$. We have

$$\varphi_0(x) = \frac{2 \operatorname{sgn} x}{\pi g(x)} \left[M_0 - \int_k^1 \frac{g(\xi) f_0'(\xi) \xi}{\xi^2 - x^2}\, d\xi \right] \tag{1.3}$$

$$M_0 = \int_k^1 \varphi_0(x)\, x\, dx = \int_k^1 \left[\frac{E(k)}{K(k)} - 1 + x^2 \right] \frac{f_0(x)}{g(x)}\, dx$$